CHAPTER 2 PROBLEMS AND EXERCISES SOLUTIONS

Table 1 T300-914 CFRP unidirectional composite stiffness matrix and density

$$\mathbf{C} = \mathbf{C}' = \begin{bmatrix} 143.8 & 6.2 & 6.2 & 0 & 0 & 0 \\ 6.2 & 13.3 & 6.5 & 0 & 0 & 0 \\ 6.2 & 6.5 & 13.3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3.4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5.7 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5.7 \end{bmatrix} \text{GPa}, \qquad \rho = 1560 \text{ kg/m}^3$$

Table 2 Typical fiber and matrix properties (www.matweb.com)

	T300 carbon fiber	S-glass fiber	914 epoxy resin
Tensile modulus, GPa	231	86.9	3.90
Poisson ratio	0.20	0.20	0.41
specific density	1.76	2.49	1.29

Table 3 Typical CFRP engineering elastic properties

	E_L, GPa	E_T, GPa	G_{LT}, GPa	G_{23}, GPa	ν_{LT}
T300-914 CFRP	140.0	10.05	5.70	3.40	0.313

PROBLEM 1: FROM STIFFNESS TENSOR TO STIFFNESS MATRIX

Given the stiffness tensor $\mathbf{c}$, do the following:

(a) verify the symmetry properties of the stiffness tensor $\mathbf{c}$

(b) find the stiffness matrix $\mathbf{C}$

Numerical example: T300-914 CFRP unidirectional composite material with $\mathbf{c}$ given as

c(:,:,1,1) = 1.0e+11 * 1.4380 0 0 0 0.0620 0 0 0 0.0620	c(:,:,1,2) = 1.0e+09 * 0 5.7000 0 5.7000 0 0 0 0 0	c(:,:,1,3) = 1.0e+09 * 0 0 5.7000 0 0 0 5.7000 0 0
c(:,:,2,1) = 1.0e+09 * 0 5.7000 0 5.7000 0 0 0 0 0	c(:,:,2,2) = 1.0e+10 * 0.6200 0 0 0 1.3300 0 0 0 0.6500	c(:,:,2,3) = 1.0e+09 * 0 0 0 0 0 3.4000 0 3.4000 0
c(:,:,3,1) = 1.0e+09 * 0 0 5.7000 0 0 0 5.7000 0 0	c(:,:,3,2) = 1.0e+09 * 0 0 0 0 0 3.4000 0 3.4000 0	c(:,:,3,3) = 1.0e+10 * 0.6200 0 0 0 0.6500 0 0 0 1.3300

The units of $\mathbf{c}$ are Pa.

<u>Solution</u>:

(a) verify the symmetry properties of the stiffness tensor:

Recall Eq. (2.21) which spells out the symmetry properties of the stiffness tensor as:

$$c_{ijkl} = c_{jikl} = c_{ijlk} = c_{klij} = c_{jilk} \qquad i,j,k,l = 1,2,3 \tag{2.21}$$

There are four symmetry properties to be verified. This is done as follow:

(a1) The relation $c_{ijkl} = c_{jikl}$ is verified because all the 2×2 matrices given above are symmetric

(a2) The condition $c_{ijkl} = c_{ijlk}$ is verified because, for $k=1$, $l=2$,

$$c_{ij12} = 10^9 \begin{bmatrix} 0 & 5.7000 & 0 \\ 5.7000 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad c_{ij21} = 10^9 \begin{bmatrix} 0 & 5.7000 & 0 \\ 5.7000 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = c_{ij12}$$

Similarly, for $k=1$, $l=3$, $c_{ij13} = c_{ij31}$ and for $k=2$, $l=3$, $c_{ij23} = c_{ij32}$

(a3) The condition $c_{ijkl} = c_{klij}$ is verified as follows. Say, $k = 1$, $l = 2$; then, on the one hand, we pick up c_{ij12} directly as

$$c_{ij12} = 10^9 \begin{bmatrix} 0 & 5.7000 & 0 \\ 5.7000 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

On the other hand, we identify the elements c_{12ij} as $c_{1211} = 0$, $c_{1212} = 10^9 \times 5.7000$, $c_{1213} = 0$ $c_{1221} = 10^9 \times 5.7000$, $c_{1222} = 0$ $c_{1223} = 0$, $c_{1231} = 0$, $c_{1232} = 0$, $c_{1233} = 0$. Hence, c_{12ij} is given by

$$c_{12ij} = 10^9 \begin{bmatrix} 0 & 5.7000 & 0 \\ 5.7000 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Similarly, for $k = 1$, $l = 3$, $c_{13ij} = c_{ij13}$ and for $k = 2$, $l = 3$, $c_{23ij} = c_{ij23}$

(a4) The condition $c_{ijkl} = c_{jilk}$ is satisfied because, according to (a2), $c_{ijkl} = c_{ijlk}$ and then, according to (a1) $c_{jilk} = c_{ijlk}$; hence, $c_{jilk} = c_{ijkl}$

This concludes part (a).

(b) Find the stiffness matrix $\mathbf{C}$. To get the elements of $\mathbf{C}$, use the correspondence formulae given in Eq. (2.35), i.e.,

$$\begin{bmatrix} c_{1111} \to C_{11} & c_{1122} \to C_{12} & c_{1133} \to C_{13} & c_{1123} \to C_{14} & c_{1131} \to C_{15} & c_{1112} \to C_{16} \\ & c_{2222} \to C_{22} & c_{2233} \to C_{23} & c_{2223} \to C_{24} & c_{2231} \to C_{25} & c_{2212} \to C_{26} \\ & & c_{3333} \to C_{33} & c_{3323} \to C_{34} & c_{3331} \to C_{35} & c_{3312} \to C_{36} \\ & & & c_{2323} \to C_{44} & c_{2331} \to C_{45} & c_{2312} \to C_{46} \\ & sym. & & & c_{3131} \to C_{55} & c_{3112} \to C_{56} \\ & & & & & c_{1212} \to C_{66} \end{bmatrix}$$

These correspondence formulae use the relation between tensor notation indices and Voigt matrix notation indices given by Table 2.1, i.e., 11=1, 22=2, 33=3, 12=21=6, 13=31=5, 23=32=4

After constructing the upper triangle and the diagonal of $\mathbf{C}$, populate the lower triangle using the symmetry property, i.e., $C_{ij} = C_{ji}$, $i = 2,3$, $j = i-1,...,3$. The result is

```
C =

   1.0e+11 *

    1.4380    0.0620    0.0620         0         0         0
    0.0620    0.1330    0.0650         0         0         0
    0.0620    0.0650    0.1330         0         0         0
         0         0         0    0.0340         0         0
         0         0         0         0    0.0570         0
         0         0         0         0         0    0.0570
```
Pa

This concludes the solution to Problem 1.

PROBLEM 2: FROM STIFFNESS MATRIX TO STIFFNESS TENSOR

Given the stiffness matrix $\mathbf{C}$, find the stiffness tensor $\mathbf{c}$.
Numerical example: T300-914 CFRP unidirectional composite material, Table 1.

<u>Solution</u>

Recall Eq. (2.36) giving the stiffness matrix as

$$\mathbf{C} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \end{bmatrix} \qquad \text{(stiffness matrix)} \qquad (1)$$

Use the correspondence formulae given in Eq. (2.35), i.e.,

$$\begin{bmatrix} c_{1111} \to C_{11} & c_{1122} \to C_{12} & c_{1133} \to C_{13} & c_{1123} \to C_{14} & c_{1131} \to C_{15} & c_{1112} \to C_{16} \\ & c_{2222} \to C_{22} & c_{2233} \to C_{23} & c_{2223} \to C_{24} & c_{2231} \to C_{25} & c_{2212} \to C_{26} \\ & & c_{3333} \to C_{33} & c_{3323} \to C_{34} & c_{3331} \to C_{35} & c_{3312} \to C_{36} \\ & & & c_{2323} \to C_{44} & c_{2331} \to C_{45} & c_{2312} \to C_{46} \\ & sym. & & & c_{3131} \to C_{55} & c_{3112} \to C_{56} \\ & & & & & c_{1212} \to C_{66} \end{bmatrix} \qquad (2.35)$$

These correspondence formulae use the relation between tensor notation indices and Voigt matrix notation indices given by Table 2.1, i.e., 11=1, 22=2, 33=3, 12=21=6, 13=31=5, 23=32=4 For example, the stiffness tensor element c_{2312} is the same as the stiffness matrix element C_{46} because 23=4 and 12=6. This concept can than be extended to the 3×3 submatrices of the stiffness tensor $\mathbf{c} = c_{ijkl}$. For example, c_{ij11} submatrix of the tensor $\mathbf{c} = c_{ijkl}$ can be written as

$$c_{ij11} = \begin{bmatrix} C_{11} & C_{61} & C_{51} \\ C_{61} & C_{21} & C_{41} \\ C_{51} & C_{41} & C_{31} \end{bmatrix}$$

where the last stiffness matrix elements C_{pq} have the second index always 1 because 11=1, whereas the first index follows the sequence 11=1, 12=6, 13=5, 21=6, 22=2, 23=4, 31=5, 32=4, 33=3. This rule is extended to the other eight submatrices of the stiffness tensor $\mathbf{c} = c_{ijkl}$.

One can take advantage of the symmetry properties Eq. (2.21), i.e., $c_{ijlk} = c_{ijkl}$ to reduce the amount of work.

Upon calculation, one gets, in Pa units,

```
c(:,:,1,1) =

   1.0e+11 *

    1.4380         0         0
         0    0.0620         0
         0         0    0.0620
```

```
c(:,:,1,2) =

   1.0e+09 *

         0    5.7000         0
    5.7000         0         0
         0         0         0
```

```
c(:,:,1,3) =

   1.0e+09 *

         0         0    5.7000
         0         0         0
    5.7000         0         0
```

```
c(:,:,2,1) =

   1.0e+09 *

         0    5.7000         0
    5.7000         0         0
         0         0         0
```

```
c(:,:,2,2) =

   1.0e+10 *

    0.6200         0         0
         0    1.3300         0
         0         0    0.6500
```

```
c(:,:,2,3) =

   1.0e+09 *

         0         0         0
         0         0    3.4000
         0    3.4000         0
```

```
c(:,:,3,1) =

   1.0e+09 *

         0         0    5.7000
         0         0         0
    5.7000         0         0
```

```
c(:,:,3,2) =

   1.0e+09 *

         0         0         0
         0         0    3.4000
         0    3.4000         0
```

```
c(:,:,3,3) =

   1.0e+10 *

    0.6200         0         0
         0    0.6500         0
         0         0    1.3300
```

This concludes the solution to Problem 2.

PROBLEM 3: ISOTROPIC COMPLIANCE AND STIFFNESS MATRICES

Given the engineering material properties of an isotropic material such as aluminum and steel, do the following:

(a) find the compliance matrix $\mathbf{S}$
(b) find the stiffness matrix $\mathbf{C}$
(c) verify that the compliance matrix and stiffness matrix are in inverse relationship, i.e., $\mathbf{C}=\mathbf{S}^{-1}$

Numerical values:

	Aluminum (7075-T6)	Steel (AISI 4340 normalized)
Elastic modulus, E	71.7 GPa	205 GPa
Poisson ratio, ν	0.33	0.29

http://asm.matweb.com/search/SpecificMaterial.asp?bassnum=MA7075T6
http://asm.matweb.com/search/SpecificMaterial.asp?bassnum=M434AE

Solution:

(a) to find the compliance matrix $\mathbf{S}$, recall Eq. (2.48), i.e.,

$$\mathbf{S}^{isotropic}=\begin{bmatrix} 1/E & -\nu/E & -\nu/E & 0 & 0 & 0 \\ -\nu/E & 1/E & -\nu/E & 0 & 0 & 0 \\ -\nu/E & -\nu/E & 1/E & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/G & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/G & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/G \end{bmatrix} \tag{2.48}$$

The shear modulus G is calculated in terms of elastic modulus E and Poisson ratio ν as $G=E/2(1+\nu)$. Upon calculation, one gets, in Pa^{-1} units,

```
S aluminum =

   1.0e-10 *

    0.1395   -0.0460   -0.0460         0         0         0
   -0.0460    0.1395   -0.0460         0         0         0
   -0.0460   -0.0460    0.1395         0         0         0
         0         0         0    0.3710         0         0
         0         0         0         0    0.3710         0
         0         0         0         0         0    0.3710
```

Pa^{-1}

```
S steel =

   1.0e-10 *

    0.0488   -0.0141   -0.0141         0         0         0
   -0.0141    0.0488   -0.0141         0         0         0
   -0.0141   -0.0141    0.0488         0         0         0
         0         0         0    0.1259         0         0
         0         0         0         0    0.1259         0
         0         0         0         0         0    0.1259
```
Pa^{-1}

(b) to find the stiffness matrix $\mathbf{C}$, recall Eq. (2.47), i.e.,

$$\mathbf{C}^{isotropic} = \begin{bmatrix} \lambda+2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda+2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda+2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix} \tag{2.47}$$

The Lame constants λ, μ are calculated with Eq. (2.45), i.e.,

$$\begin{aligned} \lambda &= \frac{\nu}{(1+\nu)(1-2\nu)} E \\ \mu &= G = \frac{1}{2(1+\nu)} E \end{aligned} \qquad \text{(Lame constants)} \tag{2.45}$$

Upon calculation, one gets, in Pa units,

```
C aluminum =

   1.0e+11 *

    1.0623    0.5232    0.5232         0         0         0
    0.5232    1.0623    0.5232         0         0         0
    0.5232    0.5232    1.0623         0         0         0
         0         0         0    0.2695         0         0
         0         0         0         0    0.2695         0
         0         0         0         0         0    0.2695
```
Pa

```
C steel =

   1.0e+11 *

    2.6864    1.0973    1.0973         0         0         0
    1.0973    2.6864    1.0973         0         0         0
    1.0973    1.0973    2.6864         0         0         0
         0         0         0    0.7946         0         0
         0         0         0         0    0.7946         0
         0         0         0         0         0    0.7946
```
Pa

(c) to verify that the compliance matrix and stiffness matrix are in inverse relationship, i.e., $\mathbf{C} = \mathbf{S}^{-1}$, one performs the multiplication of the two matrices and verify that the result is the identity matrix, i.e., $\mathbf{CS} = \mathbf{I}$. Upon calculation, one gets

```
verify C*S=I for aluminum =

    1.0000   -0.0000   -0.0000         0         0         0
   -0.0000    1.0000   -0.0000         0         0         0
   -0.0000   -0.0000    1.0000         0         0         0
         0         0         0    1.0000         0         0
         0         0         0         0    1.0000         0
         0         0         0         0         0    1.0000
```

```
verify C*S=I for steel =

     1     0     0     0     0     0
     0     1     0     0     0     0
     0     0     1     0     0     0
     0     0     0     1     0     0
     0     0     0     0     1     0
     0     0     0     0     0     1
```

This concludes the solution to Problem 3.

PROBLEM 4:
ENGINEERING PROPERTIES EXTRACTED FROM STIFFNESS MATRIX

Consider a unidirectional composite material given by its stiffness matrix $\mathbf{C}$. Do the following:

(a) recall the transversely isotropic relations that must exist between certain elements of the stiffness matrix $\mathbf{C}$ and verify that they are satisfied numerically.
(b) calculate the compliance matrix $\mathbf{S}$
(c) recall the transversely isotropic relations that must exist between certain elements of the compliance matrix $\mathbf{S}$ and verify that they are satisfied numerically
(d) extract the engineering constants $E_L, E_T, G_{LT}, \nu_{LT}, \nu_{23}$
(e) calculate G_{23} from the appropriate element of the stiffness matrix $\mathbf{C}$ and verify that it is compatible with the engineering constants deduced at item (d) above.

Numerical example: T300/914 CFRP, Table 1

Solution

(a) The transversely isotropic relations that must exist between certain elements of the stiffness matrix are: $C_{22} = C_{33}$, $C_{12} = C_{13}$, $C_{44} = (C_{22} - C_{23})/2$, $C_{55} = C_{66}$.
The first, second, and fourth properties are verified by inspection since $C_{22} = C_{33} = 13.3 \text{ GPa}$, $C_{12} = C_{13} = 6.2 \text{ GPa}$, $C_{55} = C_{66} = 5.7 \text{ GPa}$. The third property is verified in the MATLAB program and yields

```
calculated C44=(C(2,2)-C(2,3))/2  and original C(4,4) =
   1.0e+09 *
    3.4000    3.4000
```

Pa

(b) The compliance matrix $\mathbf{S}$ is calculated by the inversion of the stiffness matrix $\mathbf{C}$, i.e., $\mathbf{S} = \mathbf{C}^{-1}$. Upon calculation, one gets

```
S =
   1.0e-09 *
    0.0071   -0.0022   -0.0022         0         0         0
   -0.0022    0.0995   -0.0476         0         0         0
   -0.0022   -0.0476    0.0995         0         0         0
         0         0         0    0.2941         0         0
         0         0         0         0    0.1754         0
         0         0         0         0         0    0.1754
```

Pa^{-1}

(c) The transversely isotropic relations that must exist between certain elements of the compliance matrix are: $S_{22} = S_{33}$, $S_{12} = S_{13}$, $S_{44} = 2(S_{22} - S_{23})$, $C_{55} = C_{66}$.
The first, second, and fourth properties are verified by inspection since $S_{22} = S_{33} = -0.0995 \text{ GPa}^{-1}$, $S_{12} = S_{13} = -0.0022 \text{ GPa}^{-1}$, $S_{55} = S_{66} = 0.1754 \text{ GPa}^{-1}$. The third property is verified in the MATLAB program and yields

```
calculated S44=2*(S(2,2)-S(2,3))  and original S(4,4) =
   1.0e-09 *
    0.2941    0.2941
```

Pa^{-1}

Note: the numerical verifications listed above are done in short MATLAB format. The interested reader can verify that the same is true in the long MATLAB format by uncommenting the instruction 'format long' in the MATLAB code.

(d) the engineering constants $E_L, E_T, G_{LT}, \nu_{LT}, \nu_{23}$ are calculated from the compliance matrix **S** as follows:

$E_L = 1/S(1,1)$, $E_T = 1/S(2,2)$, $G_{LT} = 1/S(5,5)$, , $\nu_{LT} = -S(1,2)\,E_L$, $G_{LT} = 1/S(5,5)$

Upon calculation, one gets

```
EL, ET, GLT =
   1.0e+11 *
    1.3992    0.1005    0.0570
```

Pa

```
nuLT,  nu23 =
    0.3131    0.4782
```

(e) $G_{23} = C(4,4)$ as well as $G_{23} = E_T/2(1+\nu_{23})$. The two values agree:

```
G23, G23_nu23, difference(G23-G23_nu23) =
   1.0e+09 *
    3.4000    3.4000   -0.0000
```

Pa

This concludes the solution to Problem 4.

PROBLEM 5:
CFRP PROPERTIES ESTIMATED FROM FIBER AND MATRIX PROPERTIES

Recall the formulae for estimating the elastic properties $E_L, E_T, \nu_{LT}, G_{LT}, G_{23}$ of a composite using the fiber and matrix properties and the volume fraction v. In addition, develop a formula for estimating the composite density ρ.

Numerical example: calculate these properties for a 60% fiber volume fraction T300-914 CFRP composite with the fiber and matrix properties given in Table 2.

Solution:
Recall Eqs. (2.80), (2.91), (2.100), (2.109), (2.111), (2.112), i.e.,

$$E_L = E_f v_f + E_m v_m \qquad \text{(longitudinal modulus)} \qquad (2.80)$$

$$E_T = \left(E_f^{-1} v_f + E_m^{-1} v_m\right)^{-1} \qquad \text{(transverse modulus)} \qquad (2.91)$$

$$\nu_{LT} = \nu_f v_f + \nu_m v_m \qquad \text{(in-plane Poisson ratio)} \qquad (2.100)$$

$$G_{LT} = \left(G_f^{-1} v_f + G_m^{-1} v_m\right)^{-1} \qquad \text{(in-plane shear modulus)} \qquad (2.109)$$

$$G_{LT} = G_m \left[\frac{1 + v_f + (1 - v_f) G_m / G_f}{1 - v_f + (1 + v_f) G_m / G_f} \right] \qquad \text{(CAM in-plane shear modulus)} \qquad (2.111)$$

$$G_{23} = G_m \frac{v_f + \eta_{23} v_m}{\eta_{23} v_m + v_f G_m / G_f}, \quad \eta_{23} = \frac{3 - 4\nu_m + G_m / G_f}{4(1 - \nu_m)} \qquad \text{(transverse shear modulus)} \qquad (2.112)$$

The composite density can be calculated from the constituent densities as

$$\rho = \rho_f v_f + \rho_m v_m \qquad \text{(composite density)}$$

For numerical calculations with given v_f, recall the volume fraction balance equation Eq. (2.72), i.e.,

$$v_f + v_m + v_v = 1 \qquad \text{(volume fraction balance equation)} \qquad (2.72)$$

Assume zero voids, $v_v = 0$; hence, the matrix volume fraction v_m can be calculated from the fiber volume fraction v_f as

$$v_m = 1 - v_f \qquad \text{(matrix volume fraction)}$$

Numerical results:

```
T300 carbon fiber 914 epoxy CFRP results
volume fraction: vf, vm =
    0.6000    0.4000
Modulus, GPa: Ef, Em, EL, ET =
  231.0000    3.9000  140.1600    9.5092
Shear modulus, GPa: Gf, Gm, GLT, GLT_CAM, G23, =
   96.2500    1.3830    3.3845    5.2500    4.7687
Poisson ratio: nu_f, nu_m, nu_LT =
    0.2000    0.4100    0.2840
density, kg/m^3: rho_f, rho_m, rho =
        1760        1290        1572
```

This concludes the solution to Problem 5.

PROBLEM 6:
GFRP PROPERTIES ESTIMATED FROM FIBER AND MATRIX PROPERTIES

Recall the formulae for estimating the elastic properties $E_L, E_T, \nu_{LT}, G_{LT}, G_{23}$ of a composite using the fiber and matrix properties and the volume fraction v. In addition, develop a formula for estimating the composite density ρ.

Numerical example: calculate these properties for a 60% fiber volume fraction S-glass-914-epoxy GFRP composite with the fiber and matrix properties given in Table 2.

Solution:
Recall Eqs. (2.80), (2.91), (2.100), (2.109), (2.111), (2.112), i.e.,

$$E_L = E_f v_f + E_m v_m \qquad \text{(longitudinal modulus)} \tag{2.80}$$

$$E_T = \left(E_f^{-1} v_f + E_m^{-1} v_m\right)^{-1} \qquad \text{(transverse modulus)} \tag{2.91}$$

$$\nu_{LT} = \nu_f v_f + \nu_m v_m \qquad \text{(in-plane Poisson ratio)} \tag{2.100}$$

$$G_{LT} = \left(G_f^{-1} v_f + G_m^{-1} v_m\right)^{-1} \qquad \text{(in-plane shear modulus)} \tag{2.109}$$

$$G_{LT}^{CAM} = G_m \left[\frac{1 + v_f + (1 - v_f) G_m / G_f}{1 - v_f + (1 + v_f) G_m / G_f} \right] \qquad \text{(CAM in-plane shear modulus)} \tag{2.111}$$

$$\begin{aligned} G_{23} &= G_m \frac{v_f + \eta_{23} v_m}{\eta_{23} v_m + v_f G_m / G_f} \\ \eta_{23} &= \frac{3 - 4\nu_m + G_m / G_f}{4(1 - \nu_m)} \end{aligned} \qquad \text{(transverse shear modulus)} \tag{2.112}$$

The composite density can be calculated from the constituent densities as

$$\rho = \rho_f v_f + \rho_m v_m \qquad \text{(composite density)}$$

For numerical calculations with given v_f, recall Eq. (2.72) given the volume fraction balance equation, i.e.,

$$v_f + v_m + v_v = 1 \qquad \text{(volume fraction balance equation)} \tag{2.72}$$

Assume zero voids; hence, the matrix volume fraction v_m can be calculated from the fiber volume fraction v_f as

$$v_m = 1 - v_f \qquad \text{(matrix volume fraction)}$$

Numerical results:

```
S-glass 914 epoxy GFRP results
volume fraction: vf, vm =
    0.6000    0.4000
Modulus, GPa: Ef, Em, EL, ET =
   86.9000    3.9000   53.7000    9.1350
Shear modulus, GPa: Gf, Gm, GLT, GLT_CAM, G23, =
   36.2083    1.3830    3.2701    4.8446    4.4538
Poisson ratio: nu_f, nu_m, nu_LT =
    0.2000    0.4100    0.2840
density, kg/m^3: rho_f, rho_m, rho =
        2490        1290        2010
```

This concludes the solution to Problem 6.

PROBLEM 7:
VOLUME FRACTION ESTIMATION FROM COMPOSITE PROPERTIES

Estimate volume fraction v_f from the composite properties knowing the fiber and matrix properties:

(a) estimate v_f from density
(b) calculate the compliance matrix $\mathbf{S}$ and the engineering elastic properties
(c) estimate v_f from elastic properties as follows
   1. from E_L, E_T
   2. from G_{LT}, G_{LT}^{CAM}
   3. from G_{23}
   4. from ν_{LT}
(d) discuss your results

Numerical example: T300-914 CFRP, Table 1.

<u>Solution</u>:

Assume zero voids; hence, the matrix volume fraction v_m can be calculated from the fiber volume fraction v_f as

$$v_m = 1 - v_f \quad \text{(matrix volume fraction)}$$

(a) the composite density is given by

$$\rho = \rho_f v_f + \rho_m v_m = \rho_f v_f + \rho_m (1 - v_f) = (\rho_f - \rho_m) v_f + \rho_m$$

Upon solution,

$$v_f^\rho = \frac{\rho - \rho_m}{\rho_f - \rho_m}$$

Numerical value:

```
vf_density =
    0.5745
```

(b) The compliance matrix $\mathbf{S}$ is the inverse of the stiffness matrix $\mathbf{C}$, i.e., $\mathbf{S} = \mathbf{C}^{-1}$.
Numerical value:

```
compliance matrix S, 1/Pa =
   1.0e-09 *
    0.0071   -0.0022   -0.0022         0         0         0
   -0.0022    0.0995   -0.0476         0         0         0
   -0.0022   -0.0476    0.0995         0         0         0
         0         0         0    0.2941         0         0
         0         0         0         0    0.1754         0
         0         0         0         0         0    0.1754
```

The engineering elastic properties $E_L, E_T, \nu_{LT}, G_{LT}, G_{23}$ are related to the elements of the compliance matrix $\mathbf{S}$ as given by Eq. (2.67), i.e.,

$$
\begin{aligned}
S_{11} &= \frac{1}{E_L} & S_{23} &= -\frac{\nu_{23}}{E_T} \\
S_{12} = S_{13} &= -\frac{\nu_{LT}}{E_L} & S_{44} &= \frac{1}{G_{23}} \\
S_{22} = S_{33} &= \frac{1}{E_T} & S_{55} = S_{66} &= \frac{1}{G_{LT}}
\end{aligned}
\qquad (2.67)
$$

Hence, the engineering elastic properties can be calculated from the compliance matrix as

$E_L = 1/S_{11}$, $E_T = 1/S_{22}$, $G_{23} = 1/S_{44}$, $G_{LT} = 1/S_{55}$, $\nu_{LT} = -S_{12}E_L$

Numerical value:

```
Moduli, GPa: EL, ET, GLT, G23 =
  139.9172   10.0520    5.7000    3.4000
```

GPa

```
nuLT =
    0.3131
```

(c1) To estimate the volume fraction from E_L, apply to E_L the same argument that was applied to ρ, i.e.,

$$v_f^{E_L} = \frac{E_L - E_m}{E_f - E_m}$$

Numerical value:

```
vf_EL =
    0.5989
```

To estimate the volume fraction from E_T, recall Eq. (2.91), i.e.,

$$E_T = \left(E_f^{-1}v_f + E_m^{-1}v_m\right)^{-1} \qquad \text{(transverse modulus)} \qquad (2.91)$$

or

$$E_T^{-1} = E_f^{-1}v_f + E_m^{-1}v_m$$

To solve, write

$$E_T^{-1} = E_f^{-1}v_f + E_m^{-1}(1 - v_f) = (E_f^{-1} - E_m^{-1})v_f + E_m^{-1}$$

Upon solution,

$$v_f^{E_T} = \frac{E_T^{-1} - E_m^{-1}}{E_f^{-1} - E_m^{-1}}$$

Numerical value:

```
vf_ET =
     0.6225
```

(c2) To estimate the volume fraction from G_{LT}, recall Eq. (2.109), i.e.

$$G_{LT} = \left(G_f^{-1} v_f + G_m^{-1} v_m\right)^{-1}$$

or

$$G_{LT}^{-1} = G_f^{-1} v_f + G_m^{-1} v_m$$

To solve, write

$$G_{LT}^{-1} = G_f^{-1} v_f + G_m^{-1}(1 - v_f) = (G_f^{-1} - G_m^{-1}) v_f + G_m^{-1}$$

Upon solution,

$$v_f^{G_{LT}} = \frac{G_{LT}^{-1} - G_m^{-1}}{G_f^{-1} - G_m^{-1}}$$

Numerical value:

```
vf_GLT =
     0.7684
```

To estimate the volume fraction from G_{LT}^{CAM} calculated with the CAM expression, recall Eq. (2.111), i.e.,

$$G_{LT}^{CAM} = G_m \left[\frac{1 + v_f + (1 - v_f) G_m / G_f}{1 - v_f + (1 + v_f) G_m / G_f} \right]$$

which can be rearranged as

$$\frac{G_{LT}^{CAM}}{G_m} = \frac{1 + v_f + (1 - v_f) G_m / G_f}{1 - v_f + (1 + v_f) G_m / G_f}$$

Addition and subtraction of numerator-denominator yields

$$\frac{G_{LT}^{CAM} + G_m}{G_{LT}^{CAM} - G_m} = \frac{1 + G_m / G_f}{v_f (1 - G_m / G_f)}$$

Upon solution,

$$v_f^{G_{LT}^{CAM}} = \frac{G_f + G_m}{G_f - G_m} \cdot \frac{G_{LT}^{CAM} + G_m}{G_{LT}^{CAM} - G_m}$$

Numerical value:

```
vf_GLT_CAM =
     0.6273
```

(c3) To estimate the volume fraction from G_{23}, recall Eq. (2.112), i.e.,

$$G_{23} = G_m \frac{v_f + \eta_{23} v_m}{\eta_{23} v_m + v_f \, G_m / G_f}, \qquad \eta_{23} = \frac{3 - 4\nu_m + G_m / G_f}{4(1-\nu_m)} \tag{2.112}$$

Division by G_m, substitution of $v_m = 1 - v_f$, and expansion one gets

$$\frac{G_{23}}{G_m}\eta_{23}(1-v_f) + \frac{G_{23}}{G_f} v_f = v_f + \eta_{23}(1-v_f)$$

Upon solution,

$$v_f^{G_{23}} = \left[1 + \frac{1 - G_{23}/G_f}{\eta_{23}(G_{23}/G_m - 1)}\right]^{-1}$$

Numerical value:

```
vf_G23 =
    0.4682
```

(c4) To estimate the volume fraction from ν_{LT}, recall Eq. (2.110), i.e.,

$$\nu_{LT} = \nu_f \, v_f + \nu_m \, v_m \tag{2.100}$$

Hence,

$$v_f^{\nu_{LT}} = \frac{\nu_{LT} - \nu_m}{\nu_f - \nu_m}$$

Numerical value:

```
vf_nuLT =
    0.4613
```

(d) Discussion:
It is apparent that not all the volume fraction estimations give the same numerical results. The following comments can be made:

(i) The volume fraction estimates from elastic constants E_L and E_T are close to 0.6, which also close to the estimate from density rho.

(ii) The volume fraction estimate from G_{LT} are around 0.77 which is far away from (i)

(iii)However, the volume fraction estimate from G_{LT}^{CAM} is 0.63 which is closer to (i) than the estimate from G_{LT}. It seems that CAM formula Eq. (2.111) is a better than the simpler Eq. (2.109) in estimating the in-plane shear modulus G_{LT}

(iv)The volume fraction estimates from G_{23} and from ν_{LT} are around 0.46, which is much smaller that (i). It seems that the estimation of G_{23} and ν_{LT} with Eqs. (2.100), (2.112), respectively, is somehow imprecise. These aspects may need to be verified by experiments.

(v) In view of (i) and (iii), the most probable volume fraction value would be 0.6. A mean volume fraction v_f^{mean} can be calculated as well as its standard deviation v_f^{std} and relative standard deviation $v_f^{\text{std\%}} = \left(v_f^{\text{mean}} \big/ v_f^{\text{std}} \right) \times 100\%$

Numerical value:

```
vf_mean =          vf_std =          vf_std% =
    0.6058             0.0243            4.0088
```

The relative standard deviation of 4% is rather small, hence the estimate of 0.6 volume fraction seems to be quite credible.

This concludes the solution to Problem 7.

PROBLEM 8: PLOT ESTIMATED ENGINEERING PROPERTIES

Plot the estimated engineering elastic properties $E_L, E_T, \nu_{LT}, G_{LT}, G_{23}$ and density ρ vs. volume fiber fraction v_f for CFRP and GFRP composites. Discuss your results.

Numerical example: fiber and matrix properties from Table 2; fiber volume fraction up to 80%.

Solution:

Recall Eqs. (2.80), (2.91), (2.100), (2.109), (2.111), (2.112), i.e.,

$$E_L = E_f v_f + E_m v_m \qquad \text{(longitudinal modulus)} \tag{2.80}$$

$$E_T = \left(E_f^{-1} v_f + E_m^{-1} v_m\right)^{-1} \qquad \text{(transverse modulus)} \tag{2.91}$$

$$\nu_{LT} = \nu_f v_f + \nu_m v_m \qquad \text{(in-plane Poisson ratio)} \tag{2.100}$$

$$G_{LT} = \left(G_f^{-1} v_f + G_m^{-1} v_m\right)^{-1} \qquad \text{(in-plane shear modulus)} \tag{2.109}$$

$$G_{LT}^{CAM} = G_m \left[\frac{1 + v_f + (1 - v_f) G_m / G_f}{1 - v_f + (1 + v_f) G_m / G_f}\right] \qquad \text{(CAM in-plane shear modulus)} \tag{2.111}$$

$$\begin{aligned} G_{23} &= G_m \frac{v_f + \eta_{23} v_m}{\eta_{23} v_m + v_f G_m / G_f} \\ \eta_{23} &= \frac{3 - 4\nu_m + G_m / G_f}{4(1 - \nu_m)} \end{aligned} \qquad \text{(transverse shear modulus)} \tag{2.112}$$

The composite density can be calculated from the constituent densities as

$$\rho = \rho_f v_f + \rho_m v_m \qquad \text{(composite density)}$$

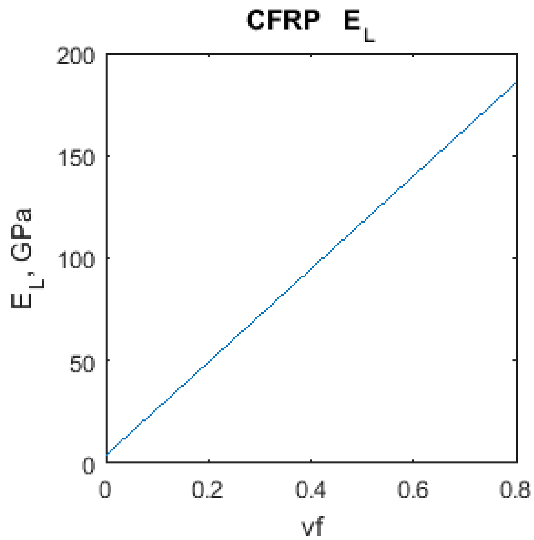
For numerical calculations with given v_f, recall Eq. (2.72) given the volume fraction balance equation, i.e.,

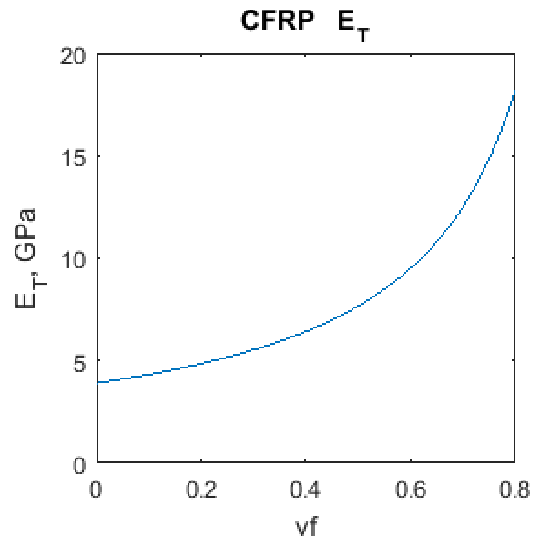
$$v_f + v_m + v_v = 1 \qquad \text{(volume fraction balance equation)} \tag{2.72}$$

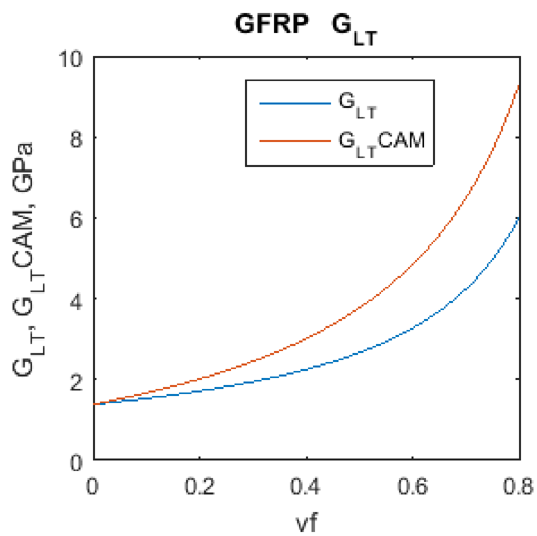
Assume zero voids; hence, the matrix volume fraction v_m can be calculated from the fiber volume fraction v_f as

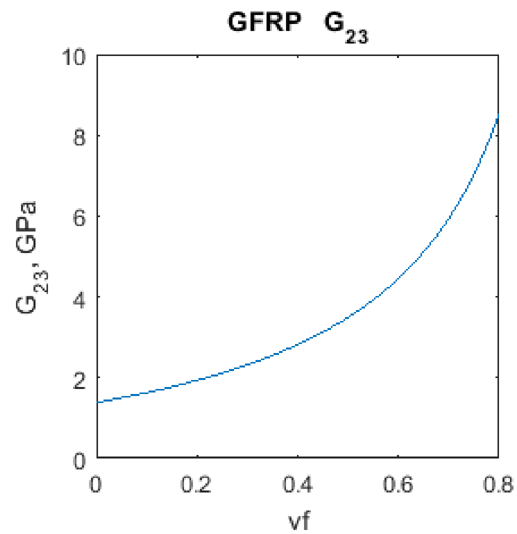
$$v_m = 1 - v_f \qquad \text{(matrix volume fraction)}$$

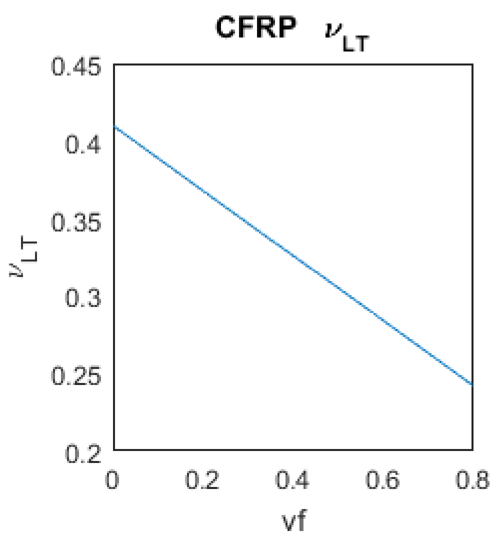
Numerical results for CFRP composite:

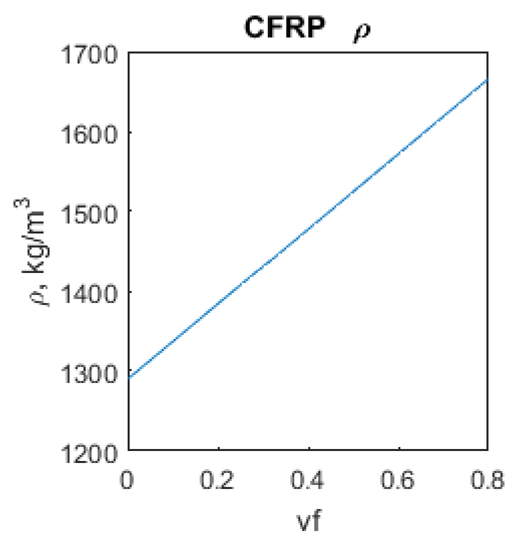




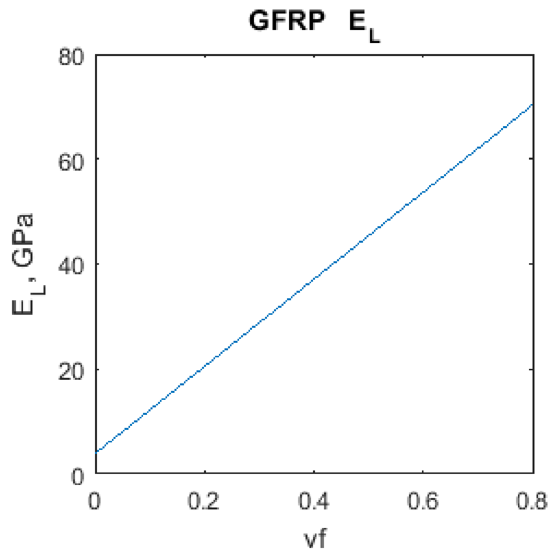


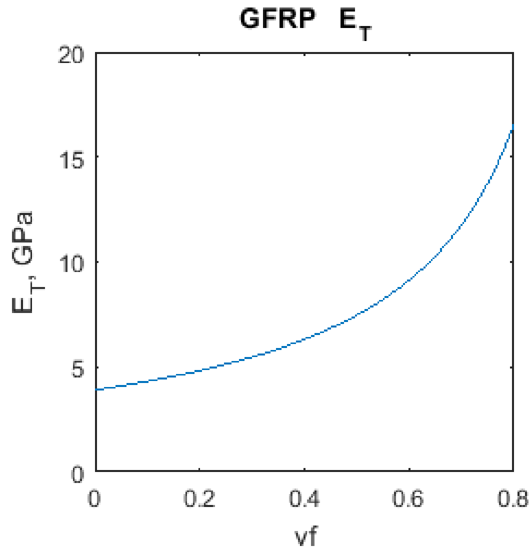


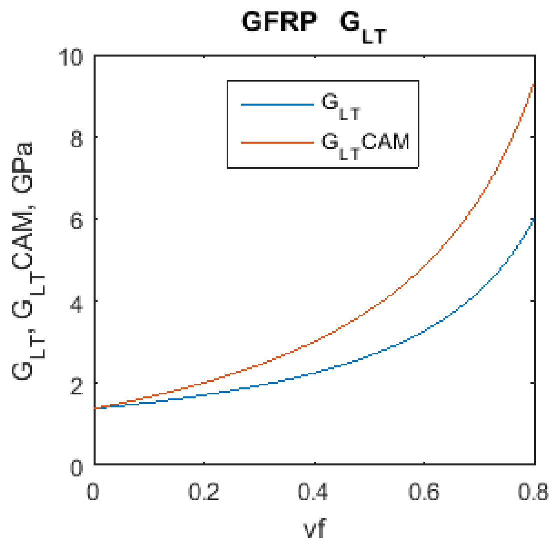


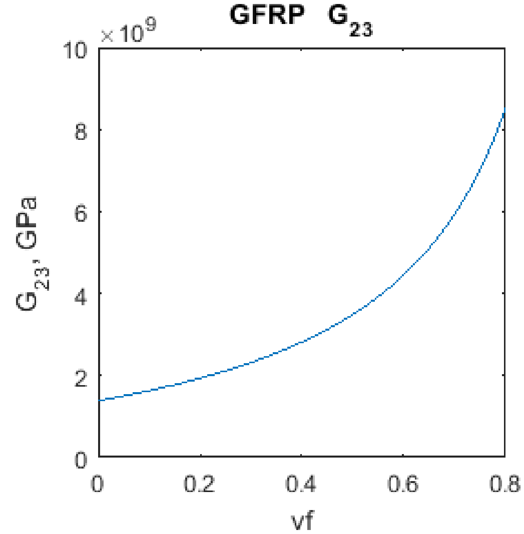


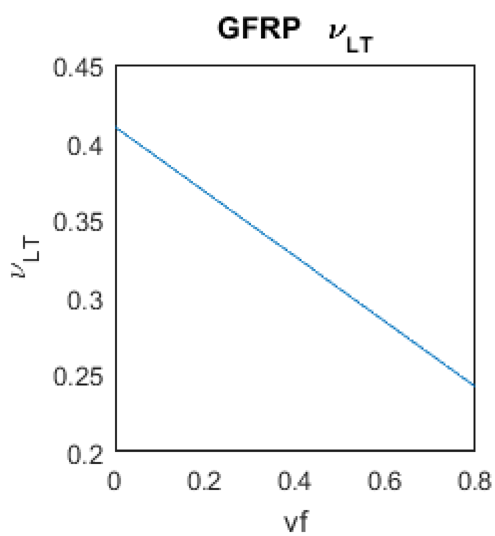
Numerical results for GFRP composite:

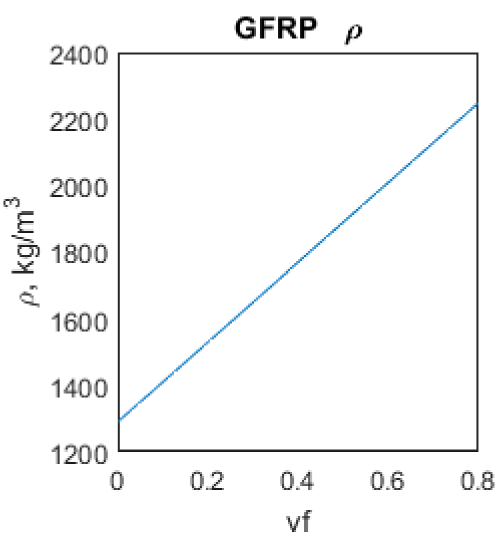












Discussion of results

a) Longitudinal modulus EL and the density rho increase linearly with volume fraction
b) In-plane Poisson ratio nuLT decreases linearly with volume fraction
c) Transverse modulus ET increases monotonically with volume fraction in an accelerating manner. It reaches about half maximum at around 60% volume fraction
d) Shear modulus GLT is estimated better by the CAM model (We base this assertion on the fact that it seems to get values closer to GLT when the CAM model is used.)
e) Shear moduli GLT and G23 increase monotonically with volume fraction in an accelerating manner. They reach about half maximum at around 60% volume fraction

This concludes the solution to Problem 8

PROBLEM 9: APPROXIMATE PROPERTY ESTIMATORS

For CFRP and GFRP composites, plot vs. volume fiber fraction v_f the approximation formulae given in Section 2.3.4.6 for estimating the engineering elastic properties in comparison with the formulae given in preceding sections. Discuss your results.

Numerical example: fiber and matrix properties from Table 2; fiber volume fraction up to 80%.

<u>Solution</u>:

Recall the approximation formulae of Eqs. (2.114), (2.115), (2.116), (2.117), in comparison with the corresponding formulae from preceding sections, i.e.,

$$E_L^a = E_f v_f \quad \text{(approx. longitudinal modulus)} \tag{2.114}$$

$$E_L = E_f v_f + E_m v_m \quad \text{(longitudinal modulus)} \tag{2.80}$$

$$E_T^a = E_m / v_m \quad \text{(approx. transverse modulus)} \tag{2.115}$$

$$E_T = \left(E_f^{-1} v_f + E_m^{-1} v_m \right)^{-1} \quad \text{(transverse modulus)} \tag{2.91}$$

$$G_{LT}^a = G_m / v_m \quad \text{(approx. in-plane shear modulus)} \tag{2.116}$$

$$G_{LT} = \left(G_f^{-1} v_f + G_m^{-1} v_m \right)^{-1} \quad \text{(in-plane shear modulus)} \tag{2.109}$$

$$G_{LT}^{CAM\,a} = G_m \frac{1+v_f}{1-v_f} \quad \text{(approx. CAM in-plane shear modulus)} \tag{2.117}$$

$$G_{LT}^{CAM} = G_m \left[\frac{1+v_f+(1-v_f)G_m/G_f}{1-v_f+(1+v_f)G_m/G_f} \right] \quad \text{(CAM in-plane shear modulus)} \tag{2.111}$$

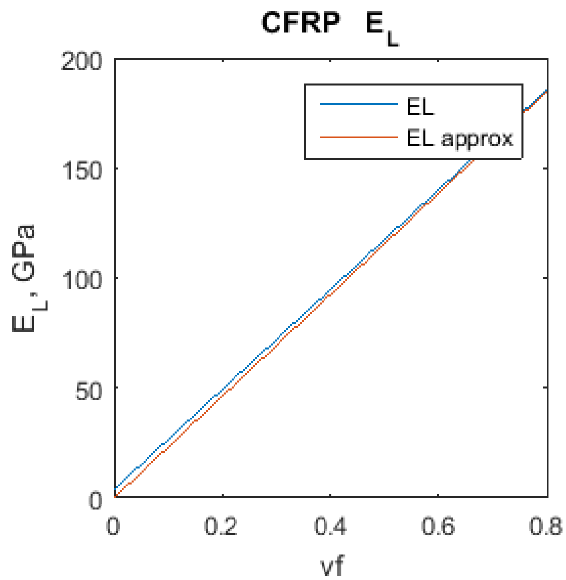
For numerical calculations with given v_f, recall Eq. (2.72) given the volume fraction balance equation, i.e.,

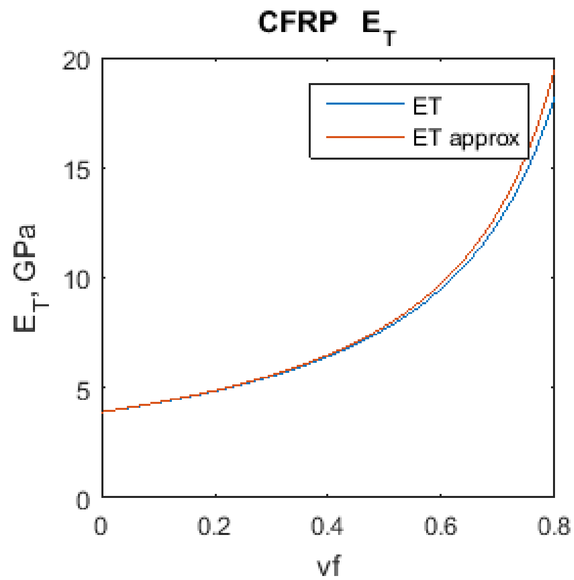
$$v_f + v_m + v_v = 1 \quad \text{(volume fraction balance equation)} \tag{2.72}$$

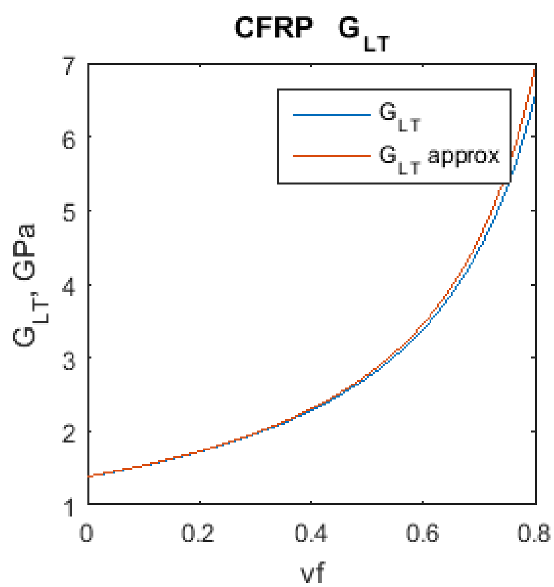
Assume zero voids; hence, the matrix volume fraction v_m can be calculated from the fiber volume fraction v_f as

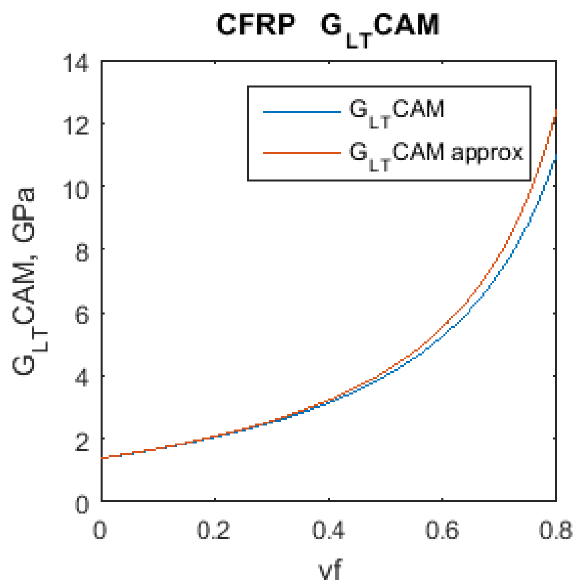
$$v_m = 1 - v_f \quad \text{(matrix volume fraction)}$$

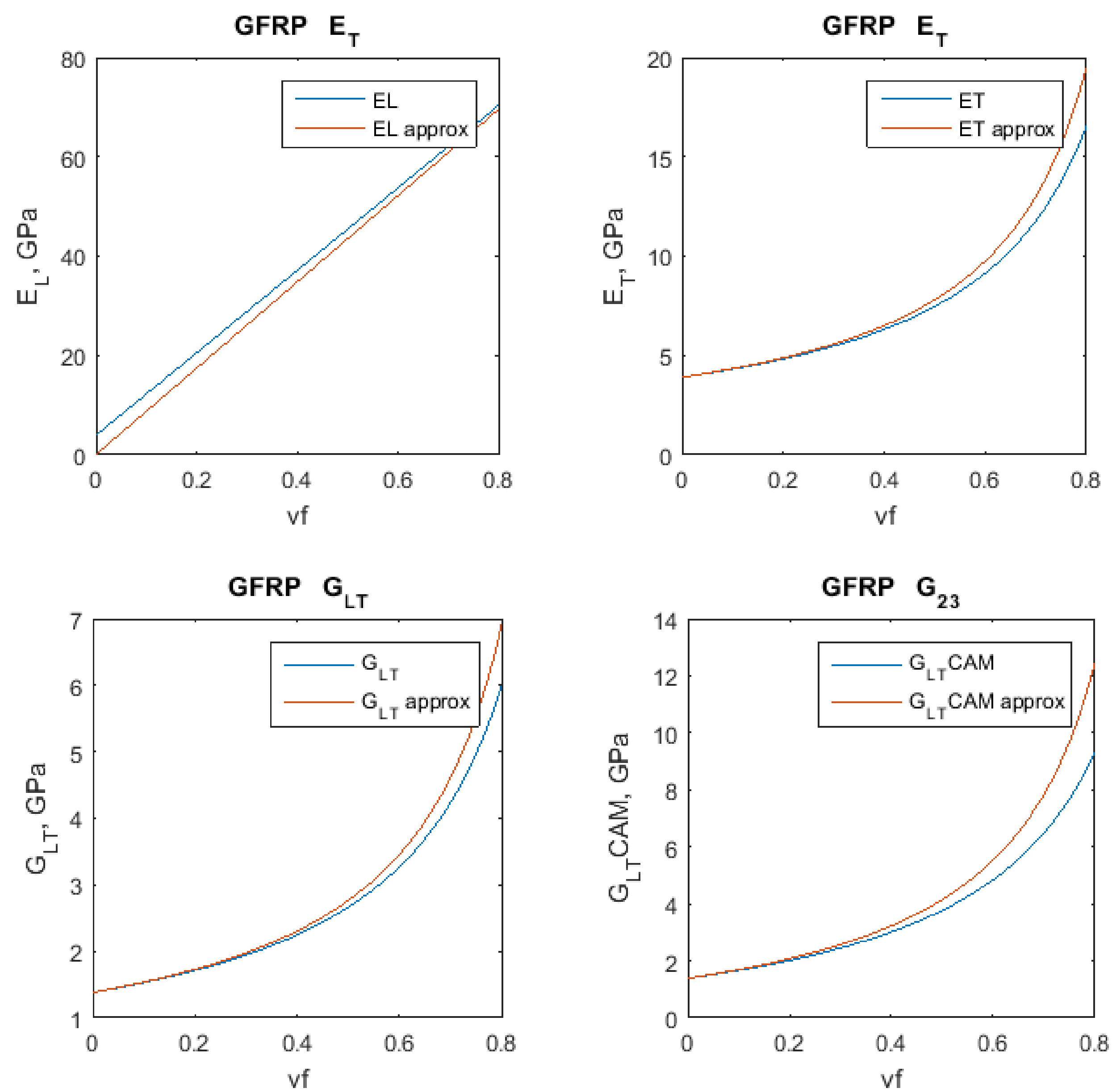
Numerical results:











<u>Discussion</u>:

- The differences between the full formulae and the approximate formulae is very small.
- These differences are larger for GFRP than for CFRP composites
- For longitudinal modulus E_L, the difference is noticed in the zero fiber volume fraction range
- For transverse and shear moduli E_T, G_{LT}, the differences are noticed in the large volume fraction range
- At a practical value of 60% fiber volume fraction, $v_f = 0.6$, the differences are almost imperceptible

This concludes the solution to Problem 9.

PROBLEM 10: ESTIMATE 3D AND 2D COMPLIANCE AND STIFFNESS MATRICES

(a) Estimate the 3D compliance and stiffness matrices from the engineering properties.
(b) Estimate the 2D compliance and stiffness matrices from the engineering properties
(c) Discuss your results

Numerical example: engineering properties of T300-914 CFRP unidirectional composite of Table 3.

Solution:

(a) Recall Eq. (2.65) that gives the 3D compliance matrix $\mathbf{S}$ in terms of engineering constants, i.e.,

$$\mathbf{S} = \begin{bmatrix} 1/E_L & -\nu_{LT}/E_L & -\nu_{LT}/E_L & 0 & 0 & 0 \\ -\nu_{LT}/E_L & 1/E_T & -\nu_{23}/E_T & 0 & 0 & 0 \\ -\nu_{LT}/E_L & -\nu_{23}/E_T & 1/E_T & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/G_{23} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/G_{LT} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/G_{LT} \end{bmatrix} \tag{2.65}$$

Invert the 3D compliance matrix $\mathbf{S}$ to get the 3D stiffness matrix $\mathbf{C}$ according to Eq. (2.69), i.e.,

$$\mathbf{C} = \mathbf{S}^{-1} \tag{2.69}$$

(b) Recall Eq. (2.121) that gives the 2D compliance matrix in terms of engineering constants, i.e.,

$$\mathbf{S}^{2\mathrm{D}} = \begin{bmatrix} 1/E_L & -\nu_{LT}/E_L & 0 \\ -\nu_{LT}/E_L & 1/E_T & 0 \\ 0 & 0 & 1/G_{LT} \end{bmatrix} \tag{2.121}$$

Invert the 2D compliance matrix $\mathbf{S}^{2\mathrm{D}}$ to get the 2D stiffness matrix $\mathbf{Q}$ according to Eq. (2.69), i.e.,

$$\mathbf{Q} = (\mathbf{S}^{2\mathrm{D}})^{-1} \tag{2.69}$$

Numerical results:

(a)

```
S =
   1.0e-09 *
    0.0071   -0.0022   -0.0022         0         0         0
   -0.0022    0.0995   -0.0476         0         0         0
   -0.0022   -0.0476    0.0995         0         0         0
         0         0         0    0.2941         0         0
         0         0         0         0    0.1754         0
         0         0         0         0         0    0.1754
```
Pa^{-1}

```
C =
   1.0e+11 *
    1.4388    0.0619    0.0619         0         0         0
    0.0619    0.1329    0.0649         0         0         0
    0.0619    0.0649    0.1329         0         0         0
         0         0         0    0.0340         0         0
         0         0         0         0    0.0570         0
         0         0         0         0         0    0.0570
```
Pa

(b)

```
S_2D =
   1.0e-09 *
    0.0071   -0.0022         0
   -0.0022    0.0995         0
         0         0    0.1754
```
Pa^{-1}

```
Q =
   1.0e+11 *
    1.4099    0.0317         0
    0.0317    0.1012         0
         0         0    0.0570
```
Pa

(c) Discussion

It is apparent that the 2D compliance matrix $\mathbf{S}^{2D}$ is a submatrix of the 3D compliance matrix $\mathbf{S}$ since the elements of $\mathbf{S}^{2D}$ can be identified among the elements of $\mathbf{S}$, both symbolically and numerically. However, the 2D stiffness matrix $\mathbf{Q}$ is not a submatrix of the 3D stiffness matrix $\mathbf{C}$ and the values of respective elements are clearly different.

This concludes the solution to Problem 10.

PROBLEM 11: ROTATE 2D COMPLIANCE AND STIFFNESS MATRICES

(a) Calculate the rotated 2D compliance matrix for a range of θ values.
(b) Calculate the rotated 2D stiffness matrix for a range of θ values.
(c) Calculate the rotated 3D stiffness matrix directly using the $\mathbf{T}$ matrix and compare the results with that the result is same with that of item (b)
(d) Discuss your results

Numerical example: T300-914 CFRP unidirectional composite and $\theta = 0°, 30°, 45°, 60°, 90°$

<u>Solution</u>:

(a) the rotated 2D compliance matrix is given by Eq. (2.155), i.e.,

$$\mathbf{S} = \mathbf{T}^t \mathbf{S}' \mathbf{T} \tag{2.155}$$

where $\mathbf{T}$ is the rotation matrix given by Eq. (2.131) i.e.,

$$\mathbf{T} = \begin{bmatrix} \cos^2\theta & \sin^2\theta & 2\sin\theta\cos\theta \\ \sin^2\theta & \cos^2\theta & -2\sin\theta\cos\theta \\ -\sin\theta\cos\theta & \sin\theta\cos\theta & \cos^2\theta - \sin^2\theta \end{bmatrix} \tag{2.131}$$

and $\mathbf{T}^t$ is its transpose of $\mathbf{T}$. The matrix $\mathbf{S}'$ is the 2D compliance matrix in material axes given by Eq. (2.121), i.e.,

$$\mathbf{S}' = \begin{bmatrix} 1/E_L & -\nu_{LT}/E_L & 0 \\ -\nu_{LT}/E_L & 1/E_T & 0 \\ 0 & 0 & 1/G_{LT} \end{bmatrix} \tag{2.121}$$

Numerical results in Pa^{-1}:
$\mathbf{S}'$ in material axes:

```
S0 =
   1.0e-09 *
     0.0071   -0.0022         0
    -0.0022    0.0995         0
          0         0    0.1754
```

$\mathbf{S}$ in rotated axes at various angles of rotation:

```
S(0 deg) =
   1.0e-09 *
     0.0071   -0.0022         0
    -0.0022    0.0995         0
          0         0    0.1754
```

```
S(30 deg) =
   1.0e-09 *
     0.0423   -0.0143   -0.0539
    -0.0143    0.0885   -0.0261
    -0.0539   -0.0261    0.1272
```

```
S(45 deg) =
   1.0e-09 *
    0.0694   -0.0183   -0.0462
   -0.0183    0.0694   -0.0462
   -0.0462   -0.0462    0.1111
S(60 deg) =
   1.0e-09 *
    0.0885   -0.0143   -0.0261
   -0.0143    0.0423   -0.0539
   -0.0261   -0.0539    0.1272
S(90 deg) =
   1.0e-09 *
    0.0995   -0.0022         0
   -0.0022    0.0071         0
         0         0    0.1754
```

(b) to get the 2D stiffness matrix $\mathbf{Q}$, invert the 2D compliance matrix $\mathbf{S}$ according to Eq. (2.124), i.e.,

$$\mathbf{Q} = \mathbf{S}^{-1} \tag{2.124}$$

Numerical results in Pa:

```
Q(0 deg) =
   1.0e+11 *
    1.4099    0.0317         0
    0.0317    0.1012         0
         0         0    0.0570
```

$\mathbf{Q}$ in rotated axes at various angles of rotation:

```
Q(30 deg) =
   1.0e+10 *
    8.5403    2.6039    4.1539
    2.6039    1.9968    1.5130
    4.1539    1.5130    2.8571
Q(45 deg) =
   1.0e+10 *
    4.5062    3.3662    3.2718
    3.3662    4.5062    3.2718
    3.2718    3.2718    3.6194
Q(60 deg) =
   1.0e+10 *
    1.9968    2.6039    1.5130
    2.6039    8.5403    4.1539
    1.5130    4.1539    2.8571
```

```
Q(90 deg) =
   1.0e+11 *
    0.1012    0.0317         0
    0.0317    1.4099         0
         0         0    0.0570
```

(c) Use Eq. (2.146) to calculate the rotated stiffness matrix as

$$\mathbf{Q} = \mathbf{T}^{-1}\mathbf{Q}'\,\mathbf{T}^{-t} \tag{2.146}$$

where $\mathbf{Q}'$ is the material axes stiffness and $\mathbf{T}$ is the rotation matrix. Then calculate the relative difference between the stiffness matrix values calculate here and in (b).

Numerical results, in Pa:

```
Q(30 deg) via Eq. (2.146) =
   1.0e+10 *
    8.5403    2.6039    4.1539
    2.6039    1.9968    1.5130
    4.1539    1.5130    2.8571
diff_Q(30 deg) = (inv(S) - Q via Eq. (2.146))/maxQ0 =
   1.0e-15 *
         0   -0.1082   -0.0541
    0.1353    0.0271    0.1082
         0   -0.0676    0.0271
```

```
Q(45 deg) via Eq. (2.146) =
   1.0e+10 *
    4.5062    3.3662    3.2718
    3.3662    4.5062    3.2718
    3.2718    3.2718    3.6194
diff_Q(45 deg) = (inv(S) - Q via Eq. (2.146))/maxQ0 =
   1.0e-15 *
    0.7035    0.7305    0.6493
    0.7035    0.7035    0.6493
    0.6493    0.6493    0.6493
```

```
Q(60 deg) via Eq. (2.146) =
   1.0e+10 *
    1.9968    2.6039    1.5130
    2.6039    8.5403    4.1539
    1.5130    4.1539    2.8571
diff_Q(60 deg) = (inv(S) - Q via Eq. (2.146))/maxQ0 =
   1.0e-15 *
         0         0    0.0406
   -0.2164   -0.4329   -0.2706
   -0.0947   -0.1623   -0.0541
```

```
Q(90 deg) via Eq. (2.146) =
   1.0e+11 *
    0.1012    0.0317         0
    0.0317    1.4099         0
         0         0    0.0570
diff_Q(90 deg) = (inv(S) - Q via Eq. (2.146))/maxQ0 =
   1.0e-15 *
         0         0         0
    0.0034    0.2164         0
         0         0         0
```

(d) Discussion of results

- The compliance matrix for $\theta = 0°$ is identical to the original compliance matrix in material axes as expected
- The compliance matrix at $\theta = 90°$ correspond to the compliance matrix at $\theta = 0°$ having the 1 and 2 axes interchanged; same for the stiffness matrix
- At $\theta = 45°$, the (1,3) and (2,3) elements are identical since the corresponding sine and cosine terms are equal.
- the (1,3) and (2,3) elements at $\theta = 60°$ are mirror image of those at $\theta = 30°$ as expected

Similar observations can be made about the stiffness matrix.

- the relative difference between the stiffness matrix values calculated at item (b) vs. item (c) is or order 1.0e-15, which is the machine precision. Hence, we conclude that both routes give same answer, but route (b) is faster because route (c) requires the an additional matrix inversion operation.

This concludes the solution to Problem 11.

PROBLEM 12: PLOT ELEMENTS OF THE ROTATED 2D STIFFNESS MATRIX

Plot the variation with θ of $Q(1,1)$, Q(2,2), $Q(3,3)$, $Q(1,3)$, $Q(2,3)$. Discuss your results

Numerical example: T300-914 CFRP unidirectional composite: $\theta = 0°,\ldots,90°$

<u>Solution</u>:

The rotated 2D compliance matrix is given by Eq. (2.155), i.e.,

$$\mathbf{S} = \mathbf{T}^t \mathbf{S}' \mathbf{T} \tag{2.155}$$

where $\mathbf{T}$ and $\mathbf{T}^t$ are the rotation matrix Eq. (2.131) and its transpose, respectively, i.e.,

$$\mathbf{T} = \begin{bmatrix} \cos^2\theta & \sin^2\theta & 2\sin\theta\cos\theta \\ \sin^2\theta & \cos^2\theta & -2\sin\theta\cos\theta \\ -\sin\theta\cos\theta & \sin\theta\cos\theta & \cos^2\theta - \sin^2\theta \end{bmatrix} \tag{2.131}$$

whereas $\mathbf{S}'$ is the 2D compliance matrix in original axes given by Eq. (2.65), i.e.,

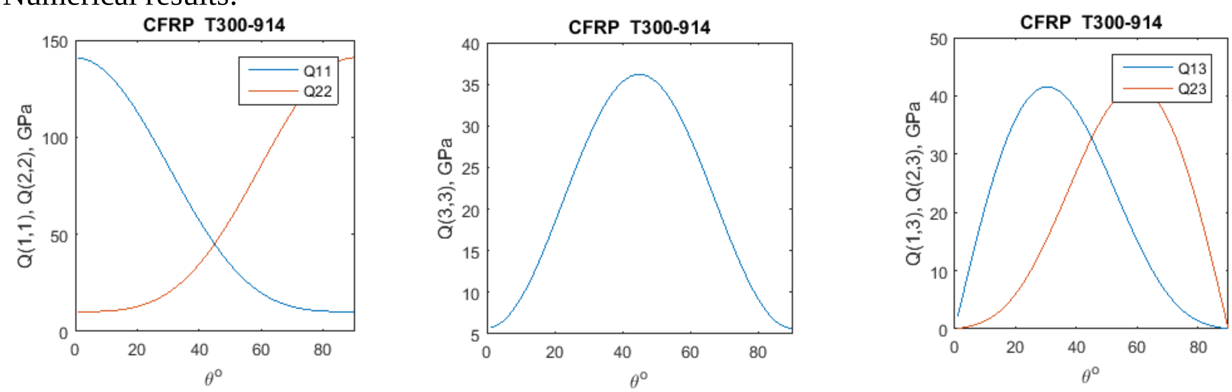
$$\mathbf{S}' = \begin{bmatrix} 1/E_L & -\nu_{LT}/E_L & 0 \\ -\nu_{LT}/E_L & 1/E_T & 0 \\ 0 & 0 & 1/G_{LT} \end{bmatrix} \tag{2.121}$$

The 2D stiffness matrix $\mathbf{Q}$, is the inverse of the 2D compliance matrix $\mathbf{S}$ according to Eq. (2.69), i.e.,

$$\mathbf{Q} = \mathbf{S}^{-1} \tag{2.69}$$

These calculations are done and stored for a range of θ. Then, the elements $Q(1,1)$, Q(2,2), $Q(3,3)$, $Q(1,3)$, $Q(1,3)$ are retrieved and plotted.

Numerical results:



Discussion:

- The elements $Q(1,1)$ and $Q(2,2)$ vary inversely, one is maximum at θ=0°, the other at θ=90°. This corresponds to E_L and E_T switching roles.
- The shear element $Q(3,3)$ is minimum at θ=0° and 90°, but maximum at 45°. This corresponds to $Q(3,3) = G_{LT}$ at θ=0° and 90°, but taking a much larger value at 45° due to diagonal fiber position.
- The shear-axial coupling elements, $Q(1,3)$, $Q(2,3)$ are maximum at θ=30° and 60°, respectively.

This concludes the solution to Problem 12.

PROBLEM 13: ROTATE 3D COMPLIANCE AND STIFFNESS MATRICES

(a) Calculate the rotated 3D compliance matrix for a range of θ values.
(b) Use the rotated 3D compliance matrix to calculate the rotated 3D stiffness matrix
(c) Calculate the rotated 3D stiffness matrix directly using the $\mathbf{T}$ matrix and compare the results with that the result is same with that of item (b)
(d) Discuss your results

Numerical example: T300-914 CFRP unidirectional composite and $\theta = 0°, 30°, 45°, 60°, 90°$

Solution:
(a) the rotated 3D compliance matrix is given by Eq. (2.203), i.e.,

$$\mathbf{S} = \mathbf{T}^t \mathbf{S}' \mathbf{T} \tag{2.203}$$

where $\mathbf{T}$ is the 3D rotation matrix given by Eq. (2.179) i.e.,

$$\mathbf{T} = \begin{bmatrix} \cos^2\theta & \sin^2\theta & 0 & 0 & 0 & 2\sin\theta\cos\theta \\ \sin^2\theta & \cos^2\theta & 0 & 0 & 0 & -2\sin\theta\cos\theta \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\theta & -\sin\theta & 0 \\ 0 & 0 & 0 & \sin\theta & \cos\theta & 0 \\ -\sin\theta\cos\theta & \sin\theta\cos\theta & 0 & 0 & 0 & \cos^2\theta - \sin^2\theta \end{bmatrix} \tag{2.179}$$

and $\mathbf{T}^t$ is its transpose of $\mathbf{T}$. The matrix $\mathbf{S}'$ is the 3D compliance matrix in material axes given by Eq. (2.165), i.e.,

$$\mathbf{S} = \begin{bmatrix} 1/E_L & -\nu_{LT}/E_L & -\nu_{LT}/E_L & 0 & 0 & 0 \\ -\nu_{LT}/E_L & 1/E_T & -\nu_{23}/E_T & 0 & 0 & 0 \\ -\nu_{LT}/E_L & -\nu_{23}/E_T & 1/E_T & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/G_{23} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/G_{LT} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/G_{LT} \end{bmatrix} \tag{2.165}$$

Numerical results in Pa^{-1}:
$\mathbf{S}'$ in material axes:

```
S0 =
   1.0e-09 *
    0.0071   -0.0022   -0.0022         0         0         0
   -0.0022    0.0995   -0.0476         0         0         0
   -0.0022   -0.0476    0.0995         0         0         0
         0         0         0    0.2941         0         0
         0         0         0         0    0.1754         0
         0         0         0         0         0    0.1754
```

S in rotated axes at various angles of rotation:

```
S(0 deg) =
   1.0e-09 *
    0.0071   -0.0022   -0.0022         0         0         0
   -0.0022    0.0995   -0.0476         0         0         0
   -0.0022   -0.0476    0.0995         0         0         0
         0         0         0    0.2941         0         0
         0         0         0         0    0.1754         0
         0         0         0         0         0    0.1754

S(30 deg) =
   1.0e-09 *
    0.0423   -0.0143   -0.0136         0         0   -0.0539
   -0.0143    0.0885   -0.0362         0         0   -0.0261
   -0.0136   -0.0362    0.0995         0         0    0.0392
         0         0         0    0.2644   -0.0514         0
         0         0         0   -0.0514    0.2051         0
   -0.0539   -0.0261    0.0392         0         0    0.1272

S(45 deg) =
   1.0e-09 *
    0.0694   -0.0183   -0.0249         0         0   -0.0462
   -0.0183    0.0694   -0.0249         0         0   -0.0462
   -0.0249   -0.0249    0.0995         0         0    0.0453
         0         0         0    0.2348   -0.0593         0
         0         0         0   -0.0593    0.2348         0
   -0.0462   -0.0462    0.0453         0         0    0.1111

S(60 deg) =
   1.0e-09 *
    0.0885   -0.0143   -0.0362         0         0   -0.0261
   -0.0143    0.0423   -0.0136         0         0   -0.0539
   -0.0362   -0.0136    0.0995         0         0    0.0392
         0         0         0    0.2051   -0.0514         0
         0         0         0   -0.0514    0.2644         0
   -0.0261   -0.0539    0.0392         0         0    0.1272

S(90 deg) =
   1.0e-09 *
    0.0995   -0.0022   -0.0476         0         0         0
   -0.0022    0.0071   -0.0022         0         0         0
   -0.0476   -0.0022    0.0995         0         0         0
         0         0         0    0.1754         0         0
         0         0         0         0    0.2941         0
         0         0         0         0         0    0.1754
```

(b) to get the 3D stiffness matrix $\mathbf{C}$, invert the 3D compliance matrix $\mathbf{S}$ according to Eq. (2.69), i.e.,

$$\mathbf{Q} = \mathbf{S}^{-1} \tag{2.69}$$

Numerical results in Pa:

```
C=inv(S)(0 deg) =
   1.0e+11 *
    1.4388    0.0619    0.0619         0         0         0
    0.0619    0.1329    0.0649         0         0         0
    0.0619    0.0649    0.1329         0         0         0
         0         0         0    0.0340         0         0
         0         0         0         0    0.0570         0
         0         0         0         0         0    0.0570
```

```
C=inv(S)(30 deg) =
   1.0e+10 *
    8.8358    2.9064    0.6267         0         0    4.1477
    2.9064    2.3066    0.6417         0         0    1.5067
    0.6267    0.6417    1.3292         0         0   -0.0130
         0         0         0    0.3975    0.0996         0
         0         0         0    0.0996    0.5125         0
    4.1477    1.5067   -0.0130         0         0    2.8572
```

```
C=inv(S)(45 deg) =
   1.0e+10 *
    4.8088    3.6688    0.6342         0         0    3.2646
    3.6688    4.8088    0.6342         0         0    3.2646
    0.6342    0.6342    1.3292         0         0   -0.0150
         0         0         0    0.4550    0.1150         0
         0         0         0    0.1150    0.4550         0
    3.2646    3.2646   -0.0150         0         0    3.6196
```

```
C=inv(S)(60 deg) =
   1.0e+10 *
    2.3066    2.9064    0.6417         0         0    1.5067
    2.9064    8.8358    0.6267         0         0    4.1477
    0.6417    0.6267    1.3292         0         0   -0.0130
         0         0         0    0.5125    0.0996         0
         0         0         0    0.0996    0.3975         0
    1.5067    4.1477   -0.0130         0         0    2.8572
```

```
C=inv(S)(90 deg) =
   1.0e+11 *
    0.1329    0.0619    0.0649         0         0         0
    0.0619    1.4388    0.0619         0         0         0
    0.0649    0.0619    0.1329         0         0         0
         0         0         0    0.0570         0         0
         0         0         0         0    0.0340         0
         0         0         0         0         0    0.0570
```

(c) Use Eq. (2.193) to calculate the rotated stiffness matrix as

$$\mathbf{C} = \mathbf{T}^{-1}\mathbf{C}'\mathbf{T}^{-t} \tag{2.193}$$

where $\mathbf{C}'$ is the material axes stiffness and $\mathbf{T}$ is the rotation matrix. Then calculate the relative difference between the stiffness matrix values calculate here and in (b).

Numerical results, in Pa:

```
C(30 deg) via Eq. (2.193) =
   1.0e+10 *
    8.8358    2.9064    0.6267         0         0    4.1477
    2.9064    2.3066    0.6417         0         0    1.5067
    0.6267    0.6417    1.3292         0         0   -0.0130
         0         0         0    0.3975    0.0996         0
         0         0         0    0.0996    0.5125         0
    4.1477    1.5067   -0.0130         0         0    2.8572
diff_C(30 deg) = (inv(S) - C via Eq. (2.193))/maxC0 =
   1.0e-15 *
         0   -0.1326    0.0066         0         0         0
    0.1326    0.0530         0         0         0    0.1061
    0.0331    0.0265    0.0133         0         0    0.0257
         0         0         0    0.0099   -0.0017         0
         0         0         0   -0.0017         0         0
         0   -0.0530   -0.0047         0         0         0

C(45 deg) via Eq. (2.193) =
   1.0e+10 *
    4.8088    3.6688    0.6342         0         0    3.2646
    3.6688    4.8088    0.6342         0         0    3.2646
    0.6342    0.6342    1.3292         0         0   -0.0150
         0         0         0    0.4550    0.1150         0
         0         0         0    0.1150    0.4550         0
    3.2646    3.2646   -0.0150         0         0    3.6196
```

```
diff_C(45 deg) = (inv(S) - C via Eq. (2.193))/maxC0 =
   1.0e-15 *
    0.6363    0.6363    0.0795         0         0    0.5568
    0.6894    0.5833    0.0729         0         0    0.4772
    0.0663    0.0663    0.0133         0         0    0.0557
         0         0         0    0.0133         0         0
         0         0         0   -0.0017    0.0133         0
    0.6363    0.5833    0.0590         0         0    0.4772

C(60 deg) via Eq. (2.193) =
   1.0e+10 *
    2.3066    2.9064    0.6417         0         0    1.5067
    2.9064    8.8358    0.6267         0         0    4.1477
    0.6417    0.6267    1.3292         0         0   -0.0130
         0         0         0    0.5125    0.0996         0
         0         0         0    0.0996    0.3975         0
    1.5067    4.1477   -0.0130         0         0    2.8572
diff_C(60 deg) = (inv(S) - C via Eq. (2.193))/maxC0 =
   1.0e-15 *
    0.0530    0.0530    0.0133         0         0    0.0133
   -0.1856   -0.5303    0.0133         0         0   -0.2651
    0.0265    0.0398    0.0133         0         0    0.0257
         0         0         0         0   -0.0025         0
         0         0         0   -0.0017    0.0099         0
   -0.0663   -0.1591    0.0083         0         0   -0.0795

C(90 deg) via Eq. (2.193) =
   1.0e+11 *
    0.1329    0.0619    0.0649         0         0         0
    0.0619    1.4388    0.0619         0         0         0
    0.0649    0.0619    0.1329         0         0         0
         0         0         0    0.0570         0         0
         0         0         0         0    0.0340         0
         0         0         0         0         0    0.0570
diff_C(90 deg) = (inv(S) - C via Eq. (2.193))/maxC0 =
   1.0e-15 *
         0         0         0         0         0         0
         0    0.2121    0.0133         0         0         0
   -0.0066    0.0066         0         0         0         0
         0         0         0         0         0         0
         0         0         0         0         0         0
         0         0         0         0         0         0
```

(d) Discussion of results

- The compliance matrix for $\theta = 0°$ is identical to the original compliance matrix in material axes as expected
- The compliance matrix at $\theta = 90°$ correspond to the compliance matrix at $\theta = 0°$ having the 1 and 2 axes interchanged; same for the stiffness matrix
- At $\theta = 45°$, the (1,6) and (2,6) elements are identical since the corresponding sine and cosine terms are equal.
- the (1,6) and (2,6) elements at $\theta = 60°$ are mirror image of those at $\theta = 30°$ as expected
- the (4,4) and (5,5) elements at $\theta = 60°$ are mirror image of those at $\theta = 30°$ as expected

Similar observations can be made about the stiffness matrix.

- the relative difference between the stiffness matrix values calculated at item (b) vs. item (c) is or order 1.0e-15, which is the machine precision. Hence, we conclude that both routes give same answer, but route (b) is faster because route (c) requires the an additional matrix inversion operation.

This concludes the solution to Problem 13.

PROBLEM 14: PLOT ELEMENTS OF THE ROTATED 3D STIFFNESS MATRIX

Plot the variation with θ of $C(1,1)$, $C(2,2)$, $C(4,4)$, $C(5,5)$, $C(6,6)$, $C(1,6)$, $C(2,6)$. Discuss your results

Numerical example: T300-914 CFRP unidirectional composite; $\theta = 0°,\ldots,90°$

Solution:

The rotated 3D compliance matrix is given by Eq. (2.203), i.e.,

$$\mathbf{S} = \mathbf{T}^t \mathbf{S}' \mathbf{T} \tag{2.203}$$

where $\mathbf{T}$ is the 3D rotation matrix given by Eq. (2.179) i.e.,

$$\mathbf{T} = \begin{bmatrix} \cos^2\theta & \sin^2\theta & 0 & 0 & 0 & 2\sin\theta\cos\theta \\ \sin^2\theta & \cos^2\theta & 0 & 0 & 0 & -2\sin\theta\cos\theta \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\theta & -\sin\theta & 0 \\ 0 & 0 & 0 & \sin\theta & \cos\theta & 0 \\ -\sin\theta\cos\theta & \sin\theta\cos\theta & 0 & 0 & 0 & \cos^2\theta - \sin^2\theta \end{bmatrix} \tag{2.179}$$

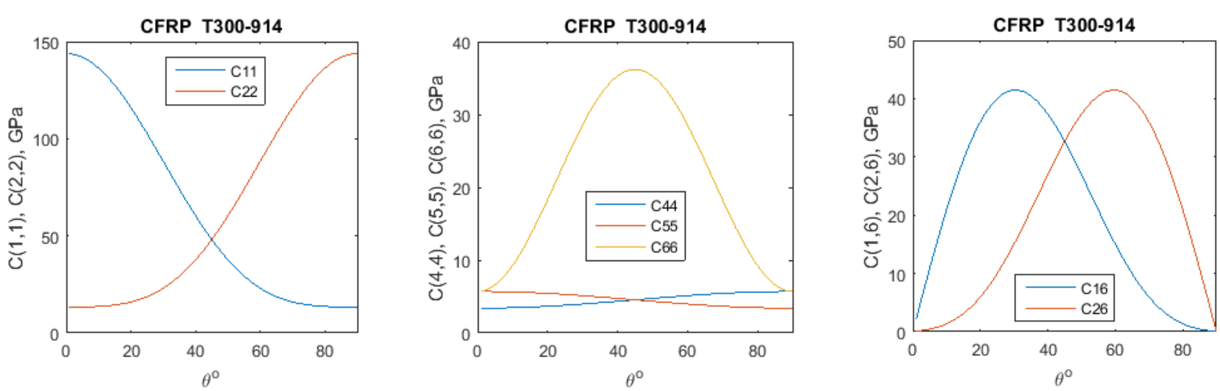
and $\mathbf{T}^t$ is its transpose of $\mathbf{T}$. The matrix $\mathbf{S}'$ is the 3D compliance matrix in material axes given by Eq. (2.165), i.e.,

$$\mathbf{S} = \begin{bmatrix} 1/E_L & -\nu_{LT}/E_L & -\nu_{LT}/E_L & 0 & 0 & 0 \\ -\nu_{LT}/E_L & 1/E_T & -\nu_{23}/E_T & 0 & 0 & 0 \\ -\nu_{LT}/E_L & -\nu_{23}/E_T & 1/E_T & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/G_{23} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/G_{LT} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/G_{LT} \end{bmatrix} \tag{2.165}$$

The 3D stiffness matrix $\mathbf{C}$ is the inverse of the 3D compliance matrix $\mathbf{S}$ according to Eq. (2.69), i.e.,

$$\mathbf{Q} = \mathbf{S}^{-1} \tag{2.69}$$

Numerical results:



Discussion:

- The elements $C(1,1)$ and $C(2,2)$ vary inversely, one is maximum at θ=0°, the other at θ=90°. This corresponds to E_L and E_T switching roles.
- The transverse shear elements $C(4,4)$ and $C(5,5)$ vary inversely, one is maximum at θ=0°, the other at θ=90°. This corresponds to G_{23} and G_{LT} switching roles.
- The in-plane shear element $C(6,6)$ is minimum at θ=0° and 90°, but maximum at 45°. This corresponds to $C\left(6,6\right)=G_{LT}$ at θ=0° and 90°, but taking a much larger value at 45° due to diagonal fiber position.
- The shear-axial coupling elements, $C(1,6)$, $C(2,6)$ are maximum at θ=30° and 60°, respectively.

This concludes the solution to Problem 14.